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The R-Functions Method in the Creep and Creep Damage Problems of Piecewise-Homogeneous Bodies of Revolution With Meridional Section of Any Shape

Sklepus S.N.[1*]

Abstract
We consider the axisymmetric problem of creep and creep damage for piecewise-homogeneous bodies of revolution with meridional section of any shape. We develop a method for the solution of the nonlinear initial boundary-value problem based on the combined application of the R -functions method and the Runge–Kutta–Merson method The structures of the solution for the main types of boundary conditions are constructed We present an example of calculation of creep, creep damage and long-term strength for a two-layer cylinder.

Keywords
Piecewise-homogeneous body, creep, creep-induced damage, R-functions method

[1] The A.N. Podgorny Institute for Mechanical Engineering Problems NAS of Ukraine, Kharkiv, Ukraine
Affiliation, Kharkiv, Ukraine
Corresponding author: snsklepus@ukr.net

1. State-of-the-Art of the Problem. Statement of the Initial Boundary-Value Problem of Creep and Creep Damage

The problems of determination of the stress-strain state and strength of piecewise homogeneous cylindrical bodies are thoroughly described in the Ukrainian and foreign literature. At the same time, the nonlinear deformation of piecewise homogeneous bodies with complex shape, in particular, the problems of creep and creep damage have not been adequately studied. This is connected with the complexity of solution of nonlinear initial boundary-value problems for piecewise homogeneous systems and with difficulties connected with the construction of constitutive equations, which must take into account various effects of deformation of contemporary materials. The analytical solution of such problems can't be obtained. There are only a few works devoted to this problem [1,2]. In [1,2] the simplified shell model was used for the formulation of the problem. In contrast to these papers we will consider the problem in the framework of the spatial formulation.

Consider a body of revolution of finite sizes referred to a cylindrical coordinate system $Orz\varphi$ at the temperature T , which consists of M components $V_1, V_2, ..., V_M$ $(V = V_1 \cup V_2 \cup ... \cup V_M)$ rigidly connected with each other. The body is under the action of external surface loads applied to a part S_p of its surface and a temperature field $T = T(r, z, t)$. The distribution of loads on S_p and given kinematically possible displacements on the surface S_u are such that the desired solution is independent of φ. The strains in the body remain small in the process of creep.

We denote by ∂V_{ab} ab the interface of the neighboring parts of the body V_a and V_b.The axis Oz coincides with the axis of revolution. The section of the body in the plane rOz has the shape of the domain Ω with boundary $\partial\Omega$. The domain Ω is the union of constituent domains Ω_k $(k = 1,...,M)$ with boundaries $\partial\Omega_k$. The rates of displacements and external loads are given on the parts of the boundary $\partial\Omega_u$ and $\partial\Omega_p$, respectively. We denote by $\partial\Omega_{ab}$ the interface of the neighboring domains $\partial\Omega_a$ and $\partial\Omega_b$. By $\partial\Omega^*_{ab}$ and $\partial\Omega^*_{ba}$, we denote the sides of the surface $\partial\Omega_{ab}$ that belong to Ω_a and Ω_b, respectively. Assume that the materials of the components of the body are isotropic and that the geometric and mechanical characteristics of each part are independent of the angular coordinate φ.

The components of the total strain rate tensor $\dot{\varepsilon}_{ij}$ consist of the components of the elastic strain rate tensors $\dot{\varepsilon}^e_{ij}$, thermal strain rate tensor $\dot{\varepsilon}^T_{ij}$, and irreversible creep strain rate tensor $\dot{p}_{ij}$

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^T_{ij} + \dot{p}_{ij}\,, \qquad (i,j=1,2,3)$$

In the cylindrical coordinate system, we have

$$\dot{\varepsilon}_r(r,z,t) = \dot{\varepsilon}^e_r(r,z,t) + \dot{\varepsilon}^T_r(r,z,t) + \dot{p}_r(r,z,t)\,, \qquad \dot{\varepsilon}_z(r,z,t) = \dot{\varepsilon}^e_z(r,z,t) + \dot{\varepsilon}^T_z(r,z,t) + \dot{p}_z(r,z,t)$$

$$\dot{\varepsilon}_\varphi(r,z,t) = \dot{\varepsilon}^e_\varphi(r,z,t) + \dot{\varepsilon}^T_\varphi(r,z,t) + \dot{p}_\varphi(r,z,t)\,, \qquad \dot{\varepsilon}_{rz}(r,z,t) = \dot{\varepsilon}^e_{rz}(r,z,t) + \dot{\varepsilon}^T_{rz}(r,z,t) + \dot{p}_{rz}(r,z,t)$$

Here, the overdot denotes the total derivative with respect to time t.

Thermal strains are calculated by the formula

$$\varepsilon^T_r = \varepsilon^T_z = \varepsilon^T_\varphi = \alpha(T - T_0),\ \varepsilon^T_{rz} = 0$$

where $T = T(r,z,t)$ is the temperature, $\alpha = \alpha(r,z,T)$ is the linear thermal expansion coefficient, and T_0 is the temperature at which stresses and strains are absent. We assume that the temperature distribution $T(r,z,t)$ is given or known from the solution of the problem of non-stationary heat conduction.

Constitutive equations of creep and creep damage growth equation of many construction materials can be described by equations [3], which can be represented as

$$\dot{p}_{kl} = \frac{3}{2}A\frac{\sigma_i^m}{(1-\psi)^n}s_{kl}\,,\ (k,l=1,2,3) \tag{1}$$

$$\dot{\psi} = B\frac{\sigma_e^k}{(1-\psi)^q} \tag{2}$$

Here $\sigma_e = \chi\sigma_i + (1-\chi)\sigma_1$ is the equivalent stress; σ_1 is the maximum principal stress; $\sigma_i = \sqrt{\frac{3}{2}s_{kl}s_{kl}}$ is the stress intensity; $s_{kl} = \sigma_{kl} - \delta_{kl}(\sigma_{kk})$ is the stress deviator, δ_{kl} is the Kronecker delta; A, B, α, m, n, k, q are material parameters; $\psi = \psi(t)$ is the scalar damage variable: $\psi(0) = 0,\ \psi(t_*) = 1$, t_* is the failure initiation time.

The boundary-value problem of creep and creep-induced damage for the axisymmetrically loaded body of revolution of finite sizes at an fixed moment of time $t \neq 0$ can be reduced to the variational problem of finding the minimum of a functional in the Lagrange form [4]. Based on functional for homogeneous body [4], we can formulate the functional in the Lagrange form for piecewise-homogeneous body defined in the space of the vectors of displacement rates

$$L(\dot{\mathbf{U}}_1, \dot{\mathbf{U}}_2, \ldots, \dot{\mathbf{U}}_M) = 0.5\sum_{k=1}^{M}\iint_{\Omega_k}\left[\lambda_{1k}\left(\dot{u}^2_{rk.r} + \dot{u}^2_{zk.z} + \dot{u}^2_{rk}r^{-2}\right) + \right.$$

$$+G_k\left(\dot{u}_{rk,z}+\dot{u}_{zk,r}\right)^2+2\lambda_k\left(\dot{u}_{rk,r}\dot{u}_{zk,z}+\dot{u}_{rk}\left(\dot{u}_{rk,r}+\dot{u}_{zk,z}\right)r^{-1}\right)\Bigg]rdrdz-$$

$$-\sum_{k=1}^{M}\iint_{\Omega_k}\left[\dot{u}_{rk,r}\dot{N}_{rk}^{f}+\dot{u}_{zk,z}\dot{N}_{zk}^{f}+\dot{u}_{rk}\dot{N}_{\varphi k}^{f}r^{-1}+\dot{N}_{rzk}^{f}\left(\dot{u}_{rk,z}+\dot{u}_{zk,r}\right)\right]rdrdz-$$

$$-\sum_{s}\int_{\partial\Omega_{ps}}\left(\dot{P}_n^0\dot{u}_{ns}+\dot{P}_\tau^0\dot{u}_{\tau s}\right)d\partial\Omega \tag{3}$$

Here $\dot{\mathbf{U}}_k(r,z,t)=\left(\dot{u}_{rk}(r,z,t),\dot{u}_{zk}(r,z,t)\right)$, $(k=1,\ldots,M)$; $\lambda_k=\dfrac{E_k\nu_k}{(1-2\nu_k)(1+\nu_k)}$, $\lambda_{1k}=\lambda_k+2G_k$, $G_k=\dfrac{E_k}{2(1+\nu_k)}$; E_k is the Young's modulus, ν_k is the Poisson's ratio of material of the k-th component; and $\dot{u}_{rk}(r,z,t)$ and $\dot{u}_{zk}(r,z,t)$ are the rates of radial and axial displacements in the k-th component, and s is the number of the component of the body to which external forces are applied; $\boldsymbol{n}$ and $\boldsymbol{\tau}$ are the external normal and tangent to $\partial\Omega$; $\dot{u}_{ns}=\dot{u}_{rs}n_1+\dot{u}_{zs}n_2$, $\dot{u}_{\tau s}=\dot{u}_{zs}n_1-\dot{u}_{rs}n_2$ are the normal component and tangential component of the vector of displacement rates; n_1 and n_2 are the direction cosines of the normal $\boldsymbol{n}$.

The rates of the "fictitious" forces in the k-th component of the body are calculated by formulas

$$\dot{N}_{rk}^{f}=\left[\lambda_{1k}\dot{e}_{rk}+\lambda_k\left(\dot{e}_{zk}+\dot{e}_{\varphi k}\right)\right],\ \dot{N}_{zk}^{f}=\left[\lambda_{1k}\dot{e}_{zk}+\lambda_k\left(\dot{e}_{rk}+\dot{e}_{\varphi k}\right)\right],$$

$$\dot{N}_{\varphi k}^{f}=\left[\lambda_{1k}\dot{e}_{\varphi k}+\lambda_k\left(\dot{e}_{rk}+\dot{e}_{zk}\right)\right],\ \dot{N}_{rzk}^{f}=2G_k\dot{e}_{rzk},$$

where

$$\dot{e}_{rk}=\dot{p}_{rk}+\dot{\varepsilon}_{rk}^{T},\ \dot{e}_{zk}=\dot{p}_{zk}+\dot{\varepsilon}_{zk}^{T},\ \dot{e}_{\varphi k}=\dot{p}_{\varphi k}+\dot{\varepsilon}_{\varphi k}^{T},\ \dot{e}_{rzk}=\dot{p}_{rzk},$$

$$\dot{\varepsilon}_{rk}^{T}=\dot{\varepsilon}_{zk}^{T}=\dot{\varepsilon}_{\varphi k}^{T}=\alpha_k\dot{T}$$

and creep strain rates $\dot{p}_{rk}$, $\dot{p}_{zk}$, $\dot{p}_{\varphi k}$, $2\dot{p}_{rzk}$ are given as some known functions of the coordinates r, z at each fixed instant of time.

The collection of the functions of the rates of displacements $\dot{u}_{rk}(r,z,t)$ and $\dot{u}_{zk}(r,z,t)$ must satisfy the following conditions:

1° be continuous together with their partial derivatives in the corresponding constituent domains Ω_k $(k=1,\ldots,M)$,

2° coincide on the boundaries of neighboring domains

$$\dot{u}_{ra}(r,z,t)=\dot{u}_{rb}(r,z,t),\ \dot{u}_{za}(r,z,t)=\dot{u}_{zb}(r,z,t)\ \text{on}\ \partial\Omega_{ab}$$

3° satisfy the following kinematic boundary conditions if the corresponding component V_l is adjacent to the boundary of the body

$$\dot{u}_{rl}(r,z,t)=\dot{f}_{rl}^{0},\ \dot{u}_{zl}(r,z,t)=\dot{f}_{zl}^{0}\ \text{ on }\ \partial\Omega_{ul} \tag{4}$$

Here, $\dot{f}_{rl}^{0}$ and $\dot{f}_{zl}^{0}$ are given functions.

On the boundaries of the neighboring domains $\partial\Omega_{ab}$, the following conditions of equilibrium must be satisfied:

$$\dot{\sigma}^a_{n_a}(r,z,t)-\dot{\sigma}^b_{n_b}(r,z,t)=0\,,\quad \dot{\tau}^a_{n_a}(r,z,t)-\dot{\tau}^b_{n_b}(r,z,t)=0\,, \tag{5}$$

where $\dot{\sigma}^a_{n_a}(r,z,t)$, $\dot{\sigma}^b_{n_b}(r,z,t)$ and $\dot{\tau}^a_{n_a}(r,z,t)$, $\dot{\tau}^b_{n_b}(r,z,t)$ are the rates of the normal and the tangential stresses that act on the surfaces $\partial\Omega^*_{ab}$ and $\partial\Omega^*_{ba}$, respectively; $\boldsymbol{n}_a$ and $\boldsymbol{n}_b$ are the outer normals to the surfaces $\partial\Omega^*_{ab}$ and $\partial\Omega^*_{ba}$, $\boldsymbol{n}_a=-\boldsymbol{n}_b$.

Conditions (5) can be obtained as conditions of stationarity or, in other words, as natural boundary conditions of functional (3) if the functions $\dot{u}_{ra}, \dot{u}_{za}, \dot{u}_{rb}$, and $\dot{u}_{zb}$ satisfy conditions **1°** and **2°**.

The main unknown problems of creep and creep damage at any point of the body can be found from the solution of the Cauchy problem with respect to time for a system of differential equations, which, for the k-th component of the body has the form

$$\begin{gathered}
\frac{du_{rk}}{dt}=\dot{u}_{rk}\,,\quad \frac{du_{zk}}{dt}=\dot{u}_{zk}\,,\\
\frac{d\varepsilon_{rk}}{dt}=\dot{u}_{rk,r}\,,\quad \frac{d\varepsilon_{zk}}{dt}=\dot{u}_{zk,z}\,,\quad \frac{d\varepsilon_{\varphi k}}{dt}=\frac{\dot{u}_{rk}}{r}\,,\quad \frac{d\gamma_{rzk}}{dt}=2\frac{d\varepsilon_{rzk}}{dt}=\dot{u}_{rk,z}+\dot{u}_{zk,r}\,,\\
\frac{d\sigma_{rk}}{dt}=\lambda_{1k}\left(\dot{\varepsilon}_{rk}-\dot{e}_{rk}\right)+\lambda_k\left(\dot{\varepsilon}_{zk}+\dot{\varepsilon}_{\varphi k}-\dot{e}_{zk}-\dot{e}_{\varphi k}\right),\quad \frac{d\sigma_{zk}}{dt}=\lambda_{1k}\left(\dot{\varepsilon}_{zk}-\dot{e}_{zk}\right)+\lambda_k\left(\dot{\varepsilon}_{rk}+\dot{\varepsilon}_{\varphi k}-\dot{e}_{rk}-\dot{e}_{\varphi k}\right),\\
\frac{d\sigma_{\varphi k}}{dt}=\lambda_{1k}\left(\dot{\varepsilon}_{\varphi k}-\dot{e}_{\varphi k}\right)+\lambda_k\left(\dot{\varepsilon}_{rk}+\dot{\varepsilon}_{zk}-\dot{e}_{rk}-\dot{e}_{zk}\right),\quad \frac{d\sigma_{rzk}}{dt}=G_k\left(\dot{\gamma}_{rzk}-2\dot{e}_{rzk}\right),\\
\frac{dp_{rk}}{dt}=\dot{p}_{rk}\,,\quad \frac{dp_{zk}}{dt}=\dot{p}_{zk}\,,\quad \frac{dp_{\varphi k}}{dt}=\dot{p}_{\varphi k}\,,\quad \frac{dp_{rzk}}{dt}=\dot{p}_{rzk}\,,\quad \frac{d\psi_k}{dt}=\dot{\psi}_k
\end{gathered} \tag{6}$$

Initial conditions at the reference state $t=0$ include natural conditions $p_{rk}=p_{zk}=p_{\varphi k}=p_{rzk}=\psi_k=0$ as well as the solution of the elastic variational problem of minimizing the functional that can be obtained from equation (3), considering the displacements u_{rk}, u_{zk} and surface forces P^0_n, P^0_τ instead of its rates and putting the "fictitious" forces: $N^f_{rk}=N^f_{zk}=N^f_{\varphi k}=N^f_{rzk}=0$.

2. Method of Solution. Structures of the Solution

We solve the initial problem (6) by the fourth-order Runge–Kutta–Merson (RKM) method of time integration with automatic time step control. The variational problems for functional (3) at times corresponding to the RKM scheme are solved by the Ritz method in combination with the R-functions method (RFM) [5]. Compared to existing numerical methods, for example FEM, RFM has a number of advantages. The R-functions method enables us to exactly take into account the geometry of the domain and the boundary conditions of the most general form. The approximate solution of the boundary-value problem is represented in the analytical form, as a structure of the solution exactly satisfying either all boundary conditions (the general structure of the solution) or a part of boundary conditions (partial structure of the solution) and is invariant with respect to the shape of the domain Ω. The structures of the solution form a base for the construction of systems of coordinate functions of variational methods.

In our case, the problem is reduced to the determination of the rates of the radial displacements $\dot{u}_{ri}$ and axial displacements $\dot{u}_{zi}$ in each of the domains Ω_i $(i=1,\ldots,M)$. Then the functions $\dot{u}_{ri}$ and $\dot{u}_{zi}$ must satisfy the equations of equilibrium written for the rates, conditions **2°** on the boundaries of neighboring components, and the kinematic boundary conditions **3°** if the corresponding component is adjacent to the boundary of the body.

Denote by Ω_n $n=(1,...,N)$ the domains adjacent to the boundary of the body. We set the rates of displacements on parts of the boundaries $\partial\Omega_{un}$ of the domains Ω_n, the stress rates on parts $\partial\Omega_{pn}$, and conditions of joining of neighboring parts on parts $\partial\Omega_{cn}$. The entire boundary of the domain Ω_n can be represented as the union of boundaries $\partial\Omega_n=\partial\Omega_{un}\cup\partial\Omega_{pn}\cup\partial\Omega_{cn}$. If only the rates of displacements or only the rates of stresses are given, then the corresponding part ($\partial\Omega_{un}$ or $\partial\Omega_{pn}$) of the boundary can be absent.

Let the boundary conditions for the rates of displacements on the part $\partial\Omega_{un}$ have the form (4). Then the partial structures of the solution for the rates of displacements $\dot{u}_{rn}$ and $\dot{u}_{zn}$ can be written in the form

$$\dot{u}_{rn}=\dot{u}_{0rn}+\dot{u}_{1rn},\ \dot{u}_{zn}=\dot{u}_{0zn}+\dot{u}_{1zn} \tag{7}$$

where $\dot{u}_{0rn}$ and $\dot{u}_{0zn}$ satisfy the inhomogeneous boundary conditions, and $\dot{u}_{1rn}$, $\dot{u}_{1zn}$ are homogeneous conditions and are determined by the following formulas

$$\dot{u}_{0rn}=\dot{f}_{rn},\ \dot{u}_{0zn}=\dot{f}_{zn} \tag{8}$$

$$\dot{u}_{1rn}=\omega_u\Phi_1+\omega^2\Phi_{10}+\omega_{ucn}^2\Phi_{1n},\ \dot{u}_{1zn}=\omega_u\Phi_2+\omega^2\Phi_{20}+\omega_{ucn}^2\Phi_{2n} \tag{9}$$

Here, $\omega(r,z)=0$ is the equation of the boundary of the body $\partial\Omega$ ($\omega>0$ inside Ω); $\omega_u(r,z)=0$ is the equation of the area $\partial\Omega_U$ ($\omega_u>0$ beyond $\partial\Omega_U$); $\omega_{ucn}(r,z)=0$ is the equation of the area $\partial\Omega_{ucn}=\partial\Omega_{un}\cup\partial\Omega_{cn}$ ($\omega_{ucn}>0$ beyond $\partial\Omega_{ucn}$). The functions $\dot{f}_{rn}$ and $\dot{f}_{zn}$ can be written in the form

$$\dot{f}_{rn}=\frac{\dot{f}^0_{rn}\omega_{pcn}}{\omega_{un}+\omega_{pcn}},\qquad \dot{f}_{zn}=\frac{\dot{f}^0_{zn}\omega_{pcn}}{\omega_{un}+\omega_{pcn}}$$

where $\omega_{pcn}(r,z)=0$ is the equation of the area $\partial\Omega_{pcn}=\partial\Omega_{pn}\cup\partial\Omega_{cn}$ ($\omega_{pcn}>0$ beyond $\partial\Omega_{pcn}$) and $\omega_{un}(r,z)=0$ is the equation of the area $\partial\Omega_{un}$ ($\omega_{un}>0$ beyond $\partial\Omega_{un}$).

The structures of the solution represented by formulas (7)–(9), satisfy exactly the kinematic boundary conditions on $\partial\Omega_u$ and the conditions of joining 2° for any choice of the undefined components $\Phi_1,\Phi_2,\Phi_{10},\Phi_{20},\Phi_{1i}$, and Φ_{2i} $(i=1,...,M)$.

If on the external boundary of some domain Ω_j, only the rates of stresses are given, i.e., in other words, the boundary $\partial\Omega_{uj}$ is absent, then, in this case, in formulas (8) and (9), we must take $\dot{u}_{0rj}\equiv 0$, $\dot{u}_{0zj}\equiv 0$, $\omega_{ucj}=\omega_{cj}$, where $\omega_{cj}(r,z)=0$ is the equation of area $\partial\Omega_{cj}$ ($\omega_{cj}>0$ beyond $\partial\Omega_{cj}$).

For the internal components Ω_l $(l=1,...,L)$, the domains Ω, on the boundaries of which only the condition of joining with the neighboring components must be satisfied, in the structures of the solution, we must take $\dot{u}_{0rl}\equiv 0$, $\dot{u}_{0zl}\equiv 0$, and $\omega_{ucl}=\omega_l$, where $\omega_l(r,z)=0$ is the equation of the boundary of the domain Ω_l ($\omega_l>0$ inside Ω_l).

The equations of the boundary of the domain Ω and its components in the structures of the solution are constructed with the help of constructive means of the R-functions theory [5].

In discretization of the boundary-value problem, the undefined components of the structures of the solution are represented in the form

$$\Phi(r,z,t) \approx \Phi_N(r,z,t) = \sum_{k=1}^{N} C_k(t)\varphi_k(r,z).$$

As $\{\varphi_k\}$, we can choose ordinary power polynomials, Chebyshev polynomials, splines, etc. [5].

3. Numerical Example for Two-Layer Cylinder

As an example, we consider the processes of creep and creep damage of a two-layer cylinder with the radius of the middle surface $R = 0.1\,\text{m}$, thickness $h = 0.01\,\text{m}$ and length $L = 0.1\,\text{m}$. Layers of cylinder have the same thickness. Both ends of the cylinder are free from external loads, but they are fixed in such a manner that the edges are restrained from radial displacements. The inner surface $r = r_{out} = R + h/2$ of the cylinder is free of loads. The outer surface $r = r_{inn} = R - h/2$ of the cylinder is loaded by a normal pressure: $P_{out} = P(z) = \frac{1}{2}P_0\left(1 + \cos\left(\frac{2\pi z}{L}\right)\right)$, where $P_0 = 20\,\text{MPa}$. The elastic constants of the materials of inner (1st) and outer (2nd) layers: $E_1 = E_2 = E = 60\,\text{GPa}$, $\nu_1 = \nu_2 = \nu = 0.35$. Here, we assume that $\alpha_1 = \alpha_2$ and $T = T_0$. Creep and creep damage growth of materials of layers constituting the cylinder are described by equations (1), (2) with values of material constants: $A_1 = 5.5 \cdot 10^{-23}\,\text{MPa}^{-(m_1+1)}\text{h}^{-1}$, $B_1 = 5.5 \cdot 10^{-24}\,\text{MPa}^{-k_1}\text{h}^{-1}$, $m_1 = 7$, $k_1 = 9$, $n_1 = q_1 = 3$, $\chi_1 = 1$; $A_2 = 13.5 \cdot 10^{-14}\,\text{MPa}^{-(m_2+1)}\text{h}^{-1}$, $B_2 = 13.5 \cdot 10^{-16}\,\text{MPa}^{-k_2}\text{h}^{-1}$, $m_2 = 3$, $k_2 = 5$, $n_2 = q_2 = 2$, $\chi_2 = 1$.

The boundary conditions on the ends of the cylinder $z = \pm L/2$ are

$$\begin{gathered} \dot{u}_{ri} = 0, \\ \dot{\sigma}_{zi} = 0, \quad (i = 1,2) \end{gathered} \tag{10}$$

On the internal surface $r = r_{inn}$ of the cylinder,

$$\dot{\sigma}_{r1} = 0, \quad \dot{\sigma}_{rz1} = 0$$

On the external surface $r = r_{out}$ of the cylinder,

$$\dot{\sigma}_{r2} = -\dot{P}_{out}, \quad \dot{\sigma}_{rz2} = 0$$

On the boundary of the layers, the following conditions of joining must be satisfied:

$$\dot{u}_{r1} = \dot{u}_{r2}, \quad \dot{u}_{z1} = \dot{u}_{z2} \tag{11}$$

Using the method of construction of the structures of the solution described in the foregoing, we can obtain partial structures satisfying conditions (10) and (11) in the next form

$$\dot{u}_{ri} = \omega_u \Phi_1 + \omega_{uci}^2 \Phi_{1i}, \quad \dot{u}_{zi} = z\left(\Phi_2 + \omega_{uci}^2 \Phi_{2i}\right), \quad (i = 1,2)$$

Here $\omega_u = \frac{1}{L}\left(\frac{L^2}{4} - z^2\right) = 0$ is equation of the part of boundary where the rates of displacement are given; $\omega_{uci} = \omega_u \wedge_0 \omega_{ci} = \omega_u + \omega_{ci} - \sqrt{\omega_u^2 + \omega_{ci}^2}$, and ω_{ci} $(i = 1,2)$ are the equations of the interfaces of the layers of cylinder

$$\omega_{c1} = R - r, \quad \omega_{c2} = r - R$$

The equation of the boundary of the domain Ω ($\omega = 0$, $\omega,_n = -1$ on $\partial\Omega$, $\omega > 0$ inside Ω) has the form

$$\omega = \omega_r \wedge_0 \omega_u = \omega_r + \omega_u - \sqrt{\omega_r^2 + \omega_u^2} = 0$$

where $\omega_r = \dfrac{(r - r_{inn})(r_{out} - r)}{r_{out} - r_{inn}}$.

In the numerical realization, the functions Φ_1, Φ_2 and Φ_{1i}, Φ_{2i} $(i = 1,2)$ are present in the form of linear combinations of Schoenberg bicubic splines [5].

Systems of spline functions were constructed on uniform rectilinear meshes. In this case, Φ_1, Φ_2 were given in the whole domain Ω, and Φ_{1i}, Φ_{2i} were given only in the corresponding subdomains Ω_i $(i = 1,2)$ with equations of the boundaries ($\omega_i = 0$, $\omega_{i,n} = -1$ on $\partial\Omega_i$, $\omega_i > 0$ inside Ω_i):

$$\omega_i = \omega_{ri} \wedge_0 \omega_u = \omega_{ri} + \omega_u - \sqrt{\omega_{ri}^2 + \omega_u^2} = 0$$

where $\omega_{ri} = \dfrac{(r - r_i)(r_{i+1} - r)}{r_{i+1} - r_i}$, $r_1 = r_{inn}$, $r_2 = R$, $r_3 = r_{out}$.

As a result of calculations, we established that the failure initiation time is $t_* = 954\,\text{h}$. The solution has been obtained with the initial value of the time step $\Delta t = 10^{-3}\,\text{h}$ and with the accuracy $\delta = 10^{-4}$. The procedure of time integration were performed up to $\psi = \psi_* = 0.9$, where ψ_* is the selected critical value of the damage variable. The damage variable reaches its critical value at the inner surface of a cylinder in its central part at the point with the coordinates $r = r_{inn} = 9.5 \cdot 10^{-2}\,\text{m}$, $z = 0$. Figures 1, 2 show the distribution of displacements of middle surface of cylinder $w = u_r(R,z,t)$ and tangential stresses σ_φ on the inner surface of cylinder along the axis Oz at different instants of time (1 - $t = 0$, 2 - $t = 500\,\text{h}$, $t = t_*$). In Fig. 3, we show growth with time of the creep damage variable ψ (a) and time variation of stress intensity σ_i $\left(\sigma_i = \sqrt{\sigma_r^2 + \sigma_z^2 + \sigma_\varphi^2 - \sigma_r\sigma_z - \sigma_r\sigma_\varphi - \sigma_\varphi\sigma_z + 3\sigma_{rz}^2}\right)$ (b) at the inner surface in the center of the cylinder.

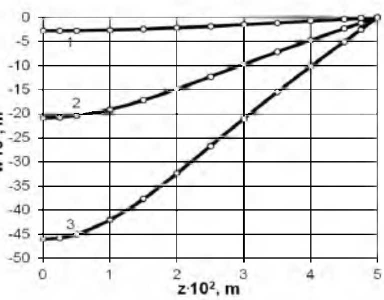

Figure 1. Distribution of displacements of middle surface of cylinder

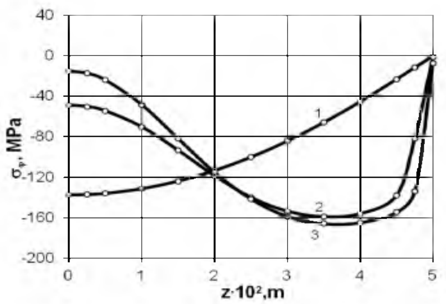

Figure 2. Distribution of tangential stesses on the inner surface of cylinder

at different instants of time at different instants of time

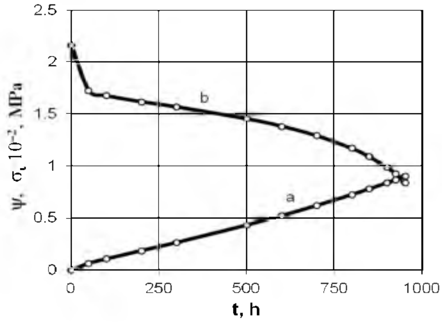

Figure 3. Growth with time of the creep damage variable (a) and time variation of stress intensity (b) at the inner surface in the center of the cylinder

During creep occurs growth of displacements and stress redistribution - maximum, at the initial time, the absolute values of the stresses decrease and the minimum stresses increase. We observe that stress intensity at the point of failure initiation with the coordinates $r = r_{inn}$, $z = 0$ relaxes during the whole creep process.

Conclusions

We develop a new numerical-analytic method for the solution of the axisymmetric initial boundary-value problem of creep and creep-induced damage for a piecewise homogeneous body of revolution with meridional section of any shape subjected to the action of force and temperature loads. The method is based on the combined use of the R-functions method and the fourth-order Runge–Kutta–Merson's method of time integration. We construct the structures of the solutions for the main types of boundary conditions. As an example, we solved the problem of creep, creep damage, and long-term strength for a two-layer cylinder.

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